A MATRIX-WEIGHTED ZETA FUNCTION OF A GRAPH

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1 Introduction

Zeta functions of graphs started from zeta functions of regular graphs by Ihara [6]. In [6], Ihara showed that their reciprocals are explicit polynomials. A zeta function of a regular graph G associated with a unitary representation of the fundamental group of G was developed by Sunada [11,12]. Hashimoto [5] generalized Ihara's result on the zeta function of a regular graph to an irregular graph, and showed that its reciprocal is again a polynomial given by a determinant containing the edge matrix. Bass [2] presented another determinant expression for the Ihara zeta function of an irregular graph by using its adjacency matrix.

Stark and Terras [10] gave an elementary proof of Bass' Theorem, and discussed three different zeta functions of any graph. Furthermore, various proofs of Bass' Theorem were given by Foata and Zeilberger [3], Kotani and Sunada [7].

As a matrix-variable zeta function of a graph, Watanabe and Fukumizu [13] defined the matrix-weighted zeta function of a graph and presented its determinant expression.

In this paper, we present a decomposition formula for the matrix-weighted zeta function of a regular covering of a graph G. Furthermore, we define a matrix-weighted L-function of G, and give a determinant expression of it. As an application, we express the matrix-weighted zeta function of a regular covering of G by a product of matrix-weighted L-functions of G.

Graphs treated here are finite. Let $G=(V(G),E(G))$ be a connected graph (possibly multiple edges and loops) with the set $V(G)$ of vertices and the set $E(G)$ of unoriented edges uv joining two vertices u and v. For $uv \in E(G)$, an arc (u,v) is the oriented edge from u to v. Set $D(G)=\{(u,v),(v,u) \mid uv \in E(G)\}$. For $e=(u,v) \in D(G)$, set $u=o(e)$ and $v=t(e)$. Furthermore, let $e^{-1}=(v,u)$ be the *inverse* of $e=(u,v)$.

A *path P of length n* in G is a sequence $P=(e_1,\cdots,e_n)$ of n arcs such that $e_i \in D(G)$, $t(e_i)=o(e_{i+1})(1 \leq i \leq n-1)$, where indices are treated *mod* n. If $e_i=(v_i,v_{i+1})$ $(1 \leq i \leq n)$,

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then we write $P=(v_1, \cdots, v_{n+1})$. Set $\mid P \mid = n$, $o(P)=o(e_1)$ and $t(P)=t(e_n)$. Also, P is called an $(o(P), t(P))$*-path*. We say that a path $P=(e_1, \cdots, e_n)$ has a *backtracking* if $e^{-1}_{i+1}=e_i$ for some $i(1 \leq i \leq n-1)$. A (v,w)-path is called a v*-cycle* (or v*-closed path*) if $v=w$. The *inverse cycle* of a cycle $C=(e_1, \cdots, e_n)$ is the cycle $C^{-1}=(e^{-1}_n, \cdots, e^{-1}_1)$.

We introduce an equivalence relation between cycles. Two cycles $C_1=(e_1, \cdots, e_m)$ and $C_2=(f_1, \cdots, f_m)$ are called *equivalent* if there exists k such that $f_j=e_{j+k}$ for all j. The inverse cycle of C is in general not equivalent to C. Let $[C]$ be the equivalence class which contains a cycle C. Let B^r be the cycle obtained by going r times around a cycle B. Such a cycle is called a *power* of B. A cycle C is *reduced* if C has no backtracking. Furthermore, a cycle C is *prime* if it is not a power of a strictly smaller cycle. Note that each equivalence class of prime, reduced cycles of a graph G corresponds to a unique conjugacy class of the fundamental group $\pi_1(G,v)$ of G at a vertex v of G.

The *Ihara zeta function* of a graph G is a function of $u \in \mathbf{C}$ with $|u|$ sufficiently small, defined by

$$\mathbf{Z}(G,u)=\mathbf{Z}_G(u)=\prod_{[C]}(1-u^{|C|})^{-1},$$

where $[C]$ runs over all equivalence classes of prime, reduced cycles of G(see [6]).

Let m be the number of edges of G. Furthermore, let two $m \times m$ matrices $\mathbf{B}=(\mathbf{B}_{e,f})_{e,f \in D(G)}$ and $\mathbf{J}_0=(\mathbf{J}_{e,f})_{e,f \in D(G)}$ be defined as follows:

$$\mathbf{B}_{e,f}=\begin{cases} 1 & \text{if } t(e)=o(f), \\ 0 & \text{otherwise,} \end{cases} \quad \mathbf{J}_{e,f}=\begin{cases} 1 & \text{if } f=e^{-1}, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\mathbf{B}-\mathbf{J}_0$ is called the *edge matrix* of G.

Theorem 1 (Hashimoto; Bass) *Let G be a connected graph with n vertices and m edges. Then the reciprocal of the Ihara zeta function of G is given by*

$$\mathbf{Z}(G,u)^{-1}=\det(\mathbf{I}_{2m}-u(\mathbf{B}-\mathbf{J}_0))=(1-u^2)^{r-1}\det(\mathbf{I}_n-u\mathbf{A}(G)+u^2(\mathbf{D}-\mathbf{I}_n)),$$

where r and $\mathbf{A}(G)$ are the Betti number and the adjacency matrix of G, respectively, and $\mathbf{D}=(d_{ij})$ is the diagonal matrix with $d_{ii}=\deg v_i$ where $V(G)=\{v_1, \cdots, v_n\}$.

Next, we state a matrix-weighted zeta function of a graph G. Let G be a connected graph with n vertices $v_1, \cdots, v_n$ and m edges, and $(a_1, \ldots, a_n) \in \mathbf{N}^n$. Set $a_{v_i}=a_i(1 \leq i \leq n)$. Then, for each $e=(v_i,v_j) \in D(G)$, let $\mathbf{w}(e)=\mathbf{w}(v_i,v_j)$ be an $a_i \times a_j$ matrix. The set $\{\mathbf{w}(e) \mid e \in D(G)\}$ is called the *matrix-weight* of G. For each cycle $C=(e_{i_1}, \ldots, e_{i_k})$, set

$$\mathbf{w}(C)=\mathbf{w}(e_{i_1})\cdots\mathbf{w}(e_{i_k}).$$

Then the *matrix-weighted zeta function* $\zeta_G(\mathbf{w})$ of G is defined by

$$\zeta_G(\mathbf{w})=\prod_{[C]}\det(\mathbf{I}-\mathbf{w}(C))^{-1},$$

where $[C]$ runs over all equivalence classes of prime, reduced cycles of G.

Let $D(G)=\{e_1, \cdots, e_m, e^{-1}_1, \cdots, e^{-1}_m\}$. Arrange arcs of G as follows: $e_1, e^{-1}_1, \cdots, e_m, e^{-1}_m$. Let

$$\mathbf{U}=\begin{bmatrix} \mathbf{w}(e_1) & & \mathbf{0} \\ & \mathbf{w}(e^{-1}_1) & \\ \mathbf{0} & & \ddots \end{bmatrix}.$$

Set $a = a_1 + \cdots + a_n$ and $b = \sum_{e \in D(G)} (a_{o(e)} + a_{t(e)})$. Furthermore, let two $b \times b$ matrices $\mathbf{B} = (\mathbf{B}_{e,f})_{e,f \in D(G)}$ and $\mathbf{J}_0 = (\mathbf{J}_{e,f})_{e,f \in D(G)}$ be defined as follows:

$$\mathbf{B}_{e,f} = \begin{cases} \mathbf{I}_{a_{t(e)}} & \text{if } t(e) = o(f), \\ \mathbf{0}_{a_{t(e)}, a_{o(f)}} & \text{otherwise,} \end{cases} \qquad \mathbf{J}_{e,f} = \begin{cases} \mathbf{I}_{a_{t(e)}} & \text{if } f = e^{-1}, \\ \mathbf{0}_{a_{t(e)}, a_{o(f)}} & \text{otherwise,} \end{cases}$$

where $\mathbf{B}_{e,f}$ and $\mathbf{J}_{e,f}$ are $a_{t(e)} \times a_{o(f)}$ matrices.

Next, we define an $a \times a$ matrix $\hat{\mathbf{A}} = \hat{\mathbf{A}}(G) = (A_{xy})$ as follows:

$$A_{xy} = \begin{cases} (\mathbf{I}_{a_x} - \mathbf{w}(x,y)\mathbf{w}(y,x))^{-1}\mathbf{w}(x,y) & \text{if } (x,y) \in D(G), \\ \mathbf{0}_{a_x, a_y} & \text{otherwise.} \end{cases}$$

Furthermore, an $a \times a$ matrix $\hat{\mathbf{D}} = \hat{\mathbf{D}}(G) = (D_{xy})$ is the diagonal matrix defined by

$$D_{xx} = \sum_{o(e)=x} \mathbf{w}(e)(\mathbf{I} - \mathbf{w}(e^{-1})\mathbf{w}(e))^{-1}\mathbf{w}(e^{-1}).$$

A determinant expression for $\zeta_G(\mathbf{w})$ was given as follows:

Theorem 2 (Watanabe and Fukumizu) *Let G be a connected graph, $a_v \in \mathbf{N}$ for each $v \in V(G)$, and $\{\mathbf{w}(e) \mid e \in D(G)\}$ a matrix-weight of G, where $\mathbf{w}(e)$ is an $a_{o(e)} \times a_{t(e)}$ matrix. Then the reciprocal of the matrix-weighted zeta function of G is given by*

$$\zeta_G(\mathbf{w})^{-1} = \det(\mathbf{I}_b - \mathbf{U}(\mathbf{B} - \mathbf{J}_0)) = \det(\mathbf{I}_a + \hat{\mathbf{D}} - \hat{\mathbf{A}}) \prod_{i=1}^{m} \det(\mathbf{I}_{a_{o(e_i)}} - \mathbf{w}(e_i)\mathbf{w}(e_i^{-1})),$$

where $n =\mid V(G) \mid$, $m =\mid E(G) \mid$ and $D(G) = \{e_1^{\pm 1}, \ldots, e_m^{\pm 1}\}$.

We use Amitsur's identity to present a determinant expression for a matrix-weighted L-function of a graph G. Foata and Zeilberger [3] gave a new proof of Bass' Theorem by using the algebra of Lyndon words. Let X be a finite nonempty set, $<$ a total order in X, and X^* the free monoid generated by X. Then the total order $<$ on X derives the lexicographic order $<^*$ on X^*. A *Lyndon word* in X is defined to a nonempty word in X^* which is prime, i.e., not the power l^r of any other word l for any $r \geq 2$, and which is also minimal in the class of its cyclic rearrangements under $<^*$. Let L denote the set of all Lyndon words in X.

Foata and Zeilberger[3] gave a short proof of Amitsur's identity [1].

Theorem 3 (Amitsur) *For square matrices $\mathbf{A}_1, \cdots, \mathbf{A}_k$,*

$$\det(\mathbf{I} - (\mathbf{A}_1 + \cdots + \mathbf{A}_k)) = \prod_{l \in L} \det(\mathbf{I} - \mathbf{A}_l),$$

where the product runs over all Lyndon words in $\{1, \cdots, k\}$, and $\mathbf{A}_l = \mathbf{A}_{i_1} \cdots \mathbf{A}_{i_p}$ for $l = i_1 \cdots i_p$.

In this paper, we define a matrix-weighted L-function of a graph G, and give a determinant expression of it.

In Section 2, we present a decomposition formula for the matrix-weighted zeta function of a regular covering of a graph G. In Section 3, we define a matrix-weighted L-function of G, and give a determinant expression of it. As a corollary, we obtain a decomposition formula for the matrix-weighted zeta function of a regular covering of G by matrix-weighted L-functions of G.

2 Zeta functions of regular coverings of graphs

Let G be a connected graph, and let $N(v) = \{w \in V(G) \mid (v, w) \in D(G)\}$ denote the neighbourhood of a vertex v in G. A graph H is called a *covering* of G with projection $\pi : H \longrightarrow G$ if there is a surjection $\pi : V(H) \longrightarrow V(G)$ such that $\pi|_{N(v')} : N(v') \longrightarrow N(v)$ is a bijection for all vertices $v \in V(G)$ and $v' \in \pi^{-1}(v)$. When a finite group Π acts on a graph G, the *quotient graph* G/Π is a graph whose vertices are the Π-orbits on $V(G)$, with two vertices adjacent in G/Π if and only if some two of their representatives are adjacent in G. A covering $\pi : H \longrightarrow G$ is said to be *regular* if there is a subgroup B of the automorphism group $AutH$ of H acting freely on H such that the quotient graph H/B is isomorphic to G.

Let G be a graph and Γ a finite group. Then a mapping $\alpha : D(G) \longrightarrow \Gamma$ is called an *ordinary voltage assignment* if $\alpha(v, u) = \alpha(u, v)^{-1}$ for each $(u, v) \in D(G)$. The pair (G, α) is called an *ordinary voltage graph*. The *derived graph* G^{α} of the ordinary voltage graph (G, α) is defined as follows: $V(G^{\alpha}) = V(G) \times \Gamma$ and $((u, h), (v, k)) \in D(G^{\alpha})$ if and only if $(u, v) \in D(G)$ and $k = h\alpha(u, v)$. The *natural projection* $\pi : G^{\alpha} \longrightarrow G$ is defined by $\pi(u, h) = u$. The graph G^{α} is called a *derived graph covering* of G with voltages in Γ or a Γ-*covering* of G. The natural projection π commutes with the right multiplication action of the $\alpha(e), e \in D(G)$ and the left action of Γ on the fibers: $g(u, h) = (u, gh), g \in \Gamma$, which is free and transitive. Thus, the Γ-covering G^{α} is a $\mid \Gamma \mid$-fold regular covering of G with covering transformation group Γ. Furthermore, every regular covering of a graph G is a Γ-covering of G for some group Γ(see [4]).

In the Γ-covering G^{α}, set $v_g = (v, g)$ and $e_g = (e, g)$, where $v \in V(G), e \in D(G), g \in \Gamma$. For $e = (u, v) \in D(G)$, the arc e_g emanates from u_g and terminates at $v_{g\alpha(e)}$. Note that $e_g^{-1} = (e^{-1})_{g\alpha(e)}$.

Let $a_v \in \mathbf{N}$ for each $v \in V(G)$, and $\{\mathbf{w}(e) \mid e \in D(G)\}$ a matrix-weight of G, where $\mathbf{w}(e)$ is an $a_{o(e)} \times a_{t(e)}$ matrix. Then we define the *matrix-weight* $\tilde{\mathbf{w}}$ of G^{α} *derived from* $\mathbf{w}$ as follows:

$$\tilde{\mathbf{w}}(u_g, v_h) := \begin{cases} \mathbf{w}(u, v) & \text{if } (u, v) \in D(G) \text{ and } h = g\alpha(u, v), \\ \mathbf{0}_{a_u, a_v} & \text{otherwise.} \end{cases}$$

Let $\mathbf{M}_1 \oplus \cdots \oplus \mathbf{M}_s$ be the block diagonal sum of square matrices $\mathbf{M}_1, \cdots, \mathbf{M}_s$. If $\mathbf{M}_1 = \mathbf{M}_2 = \cdots = \mathbf{M}_s = \mathbf{M}$, then we write $s \circ \mathbf{M} = \mathbf{M}_1 \oplus \cdots \oplus \mathbf{M}_s$.

Theorem 4 *Let G be a connected graph with n vertices and m edges, Γ a finite group, $\alpha : D(G) \longrightarrow \Gamma$ an ordinary voltage assignment, $a_v \in \mathbf{N}$ for each $v \in V(G)$, and $\{\mathbf{w}(e) \mid e \in D(G)\}$ a matrix-weight of G, where $\mathbf{w}(e)$ is an $a_{o(e)} \times a_{t(e)}$ matrix. Set $\mid \Gamma \mid = r$. Furthermore, let $\rho_1 = 1, \rho_2, \cdots, \rho_k$ be the irreducible representations of Γ, and d_i the degree of ρ_i for each i, where $d_1 = 1$. For $g \in \Gamma$, the matrix $\mathbf{A}_g = (a^{(g)}_{xy})$ is defined as follows:*

$$a^{(g)}_{xy} := \begin{cases} (\mathbf{I}_{a_x} - \mathbf{w}(x, y)\mathbf{w}(y, x))^{-1}\mathbf{w}(x, y) & \textit{if } (x, y) \in D(G) \textit{ and } \alpha(x, y) = g, \\ \mathbf{0}_{a_x, a_y} & \textit{otherwise.} \end{cases}$$

Suppose that the Γ-covering G^{α} of G is connected. Then the reciprocal of the matrix-weighted zeta function of G^{α} is

$$\zeta_{G^{\alpha}}(\tilde{\mathbf{w}})^{-1} = \prod_{i=1}^{m} \det(\mathbf{I}_{a_{o(e_i)}} - \mathbf{w}(e_i)\mathbf{w}(e_i^{-1}))^r \prod_{i=1}^{k} \det(\mathbf{I}_{ad_i} - \sum_{h \in \Gamma} \rho_i(h) \bigotimes \mathbf{A}_h + (\mathbf{I}_{d_i} \bigotimes \hat{\mathbf{D}}(G)))^{d_i},$$

where $D(G) = \{e_1^{\pm 1}, \ldots, e_m^{\pm 1}\}$.

Proof . Let $V(G) = \{v_1, \cdots, v_n\}$ and $\Gamma = \{1 = g_1, g_2, \cdots, g_r\}$. Arrange vertices of G^{α} in n blocks: $(v_1, 1), \cdots, (v_n, 1); (v_1, g_2), \cdots, (v_n, g_2); \cdots; (v_1, g_r), \cdots, (v_n, g_r)$. We consider two matrices $\hat{\mathbf{A}}(G^\alpha)$ and $\hat{\mathbf{D}}(G^\alpha)$ under this order. By Theorem 2, we have

$$\zeta_{G^\alpha}(\tilde{\mathbf{w}})^{-1} = \det(\mathbf{I}_{ar} - \hat{\mathbf{A}}(G^\alpha) + \hat{\mathbf{D}}(G^\alpha)) \prod_{i=1}^{m} \det(\mathbf{I}_{a_{o(e_i)}} - \mathbf{w}(e_i)\mathbf{w}(e_i^{-1}))^r .$$

For $h \in \Gamma$, the matrix $\mathbf{P}_h = (p_{ij}^{(h)})$ is defined as follows:

$$p_{ij}^{(h)} = \begin{cases} 1 & \text{if } g_i h = g_j, \\ 0 & \text{otherwise.} \end{cases}$$

Suppose that $p_{ij}^{(h)} = 1$, i.e., $g_j = g_i h$. Then $((u, g_i), (v, g_j)) \in D(G^\alpha)$ if and only if $(u, v) \in D(G)$ and $g_j = g_i \alpha(u, v)$, i.e., $\alpha(u, v) = g_i^{-1} g_j = g_i^{-1} g_i h = h$. Furthermore, if $((u, g_i), (v, g_j)) \in D(G^\alpha)$, then we have

$$\begin{aligned} (\hat{\mathbf{A}}(G^\alpha))_{u_{g_i}, v_{g_j}} &= (\mathbf{I} - \tilde{\mathbf{w}}(u_{g_i}, v_{g_j})\tilde{\mathbf{w}}(v_{g_j}, u_{g_i}))^{-1}\tilde{\mathbf{w}}(u_{g_i}, v_{g_j}) \\ &= (\mathbf{I} - \mathbf{w}(u, v)\mathbf{w}(v, u))^{-1}\mathbf{w}(u, v). \end{aligned}$$

Thus we have

$$\hat{\mathbf{A}}(G^\alpha) = \sum_{h \in \Gamma} \mathbf{P}_h \bigotimes \mathbf{A}_h .$$

Next, since $\tilde{\mathbf{w}}(\tilde{e}) = \mathbf{w}(e)$ and $\tilde{\mathbf{w}}(\tilde{e}^{-1}) = \mathbf{w}(e^{-1})$ for $e \in D(G)$ and $\tilde{e} \in \pi^{-1}(e)$, we have

$$\begin{aligned} (\hat{\mathbf{D}}(G^\alpha))_{u_{g_i}, u_{g_i}} &= \textstyle\sum_{o(\tilde{e}) = u_{g_i}} \tilde{\mathbf{w}}(\tilde{e})(\mathbf{I} - \tilde{\mathbf{w}}(\tilde{e}^{-1})\tilde{\mathbf{w}}(\tilde{e}))^{-1}\tilde{\mathbf{w}}(\tilde{e}^{-1}) \\ &= \textstyle\sum_{o(e) = u} \mathbf{w}(e)(\mathbf{I} - \mathbf{w}(e^{-1})\mathbf{w}(e))^{-1}\mathbf{w}(e^{-1}). \end{aligned}$$

Thus,

$$\hat{\mathbf{D}}(G^\alpha) = \mathbf{I}_r \bigotimes \hat{\mathbf{D}}(G).$$

Let ρ be the right regular representation of Γ. Furthermore, let $\rho_1 = 1, \rho_2, \cdots, \rho_k$ be the irreducible representations of Γ, and d_i the degree of ρ_i for each i, where $d_1 = 1$. Then we have $\rho(h) = \mathbf{P}_h$ for $h \in \Gamma$. Furthermore, there exists a nonsingular matrix $\mathbf{P}$ such that $\mathbf{P}^{-1}\rho(h)\mathbf{P} = (1) \oplus d_2 \circ \rho_2(h) \oplus \cdots \oplus d_k \circ \rho_k(h)$ for each $h \in \Gamma$(see [9]). Putting $\mathbf{F} = (\mathbf{P}^{-1} \bigotimes \mathbf{I})\hat{\mathbf{A}}(G^\alpha)(\mathbf{P} \bigotimes \mathbf{I})$, we have

$$\mathbf{F} = \sum_{h \in \Gamma} \{(1) \oplus d_2 \circ \rho_2(h) \oplus \cdots \oplus d_k \circ \rho_k(h)\} \bigotimes \mathbf{A}_h .$$

Note that $\hat{\mathbf{A}}(G) = \sum_{h \in \Gamma} \mathbf{A}_h$ and $1 + d_2^2 + \cdots + d_k^2 = r$. Therefore it follows that

$$\zeta_{G^\alpha}(\tilde{\mathbf{w}})^{-1} = \prod_{i=1}^{k} \{ \prod_{i=1}^{m} \det(\mathbf{I}_{a_{o(e_i)}} - \mathbf{w}(e_i)\mathbf{w}(e_i^{-1}))^{d_i} \det(\mathbf{I}_{ad_i} - \sum_{h \in \Gamma} \rho_i(h) \bigotimes \mathbf{A}_h + \mathbf{I}_{d_i} \bigotimes \hat{\mathbf{D}}(G)) \}^{d_i} .$$

Q.E.D.

3 L-functions of graphs

Let G be a connected graph with m edges, Γ a finite group, $\alpha : D(G) \longrightarrow \Gamma$ an ordinary voltage assignment, $a_v \in \mathbf{N}$ for each $v \in V(G)$, and $\{\mathbf{w}(e) \mid e \in D(G)\}$ a matrix-weight of G, where $\mathbf{w}(e)$ is an $a_{o(e)} \times a_{t(e)}$ matrix. For each path $P = (e_1, \cdots, e_r)$ of G, set $\alpha(P) =$

$\alpha(e_1)\cdots\alpha(e_r)$. This is called the *net voltage* of P. Furthermore, let ρ be a representation of Γ and d its degree.

The *matrix-weighted L-function* of G associated with ρ and α is defined by

$$\zeta_G(\mathbf{w},\rho,\alpha)=\prod_{[C]}\det(\mathbf{I}_{da_{o(C)}}-\rho(\alpha(C))\bigotimes\mathbf{w}(C))^{-1},$$

where $[C]$ runs over all equivalence classes of prime, reduced cycles of G.

Let $D(G)=\{e_1,e_1^{-1},\dots,e_m,e_m^{-1}\}$ and

$$\mathbf{U}_\rho=\begin{bmatrix}\rho(\alpha(e_1))\bigotimes\mathbf{w}(e_1) & & \mathbf{0}\\ & \rho(\alpha(e_1^{-1}))\bigotimes\mathbf{w}(e_1^{-1}) & \\ \mathbf{0} & & \ddots\end{bmatrix}.$$

Furthermore, let two $bd\times bd$ matrices $\mathbf{B}_\rho=(\mathbf{B}^{(\rho)}_{e,f})_{e,f\in D(G)}$ and $\mathbf{J}_\rho=(\mathbf{J}^{(\rho)}_{e,f})_{e,f\in D(G)}$ be defined as follows:

$$\mathbf{B}^{(\rho)}_{e,f}=\begin{cases}\mathbf{I}_d\bigotimes\mathbf{I}_{a_{t(e)}} & \text{if } t(e)=o(f),\\ \mathbf{0}_{da_{t(e)},da_{o(f)}} & \text{otherwise,}\end{cases}\quad \mathbf{J}^{(\rho)}_{e,f}=\begin{cases}\mathbf{I}_d\bigotimes\mathbf{I}_{a_{t(e)}} & \text{if } f=e^{-1},\\ \mathbf{0}_{da_{t(e)},da_{o(f)}} & \text{otherwise.}\end{cases}$$

A determinant expression for the matrix-weighted L-function of G associated with ρ and α is given as follows:

Theorem 5 *Let G be a connected graph, $a_v\in\mathbf{N}$ for each $v\in V(G)$, and $\{\mathbf{w}(e)\mid e\in D(G)\}$ a matrix-weight of G, where $\mathbf{w}(e)$ is an $a_{o(e)}\times a_{t(e)}$ matrix. Suppose that*

$$\det(\mathbf{I}_{a_{o(e)}}-\mathbf{w}(e)\mathbf{w}(e^{-1}))\neq 0$$

for each $e\in D(G)$. Then the reciprocal of the matrix-weighted L-function of G is given by

$$\zeta_G(\mathbf{w},\rho,\alpha)^{-1}=\det(\mathbf{I}_{bd}-\mathbf{U}_\rho(\mathbf{B}_\rho-\mathbf{J}_\rho))$$

$$=\prod_{i=1}^m\det(\mathbf{I}_{a_{o(e_i)}}-\mathbf{w}(e_i)\mathbf{w}(e_i^{-1}))^d\det(\mathbf{I}_{ad}+\mathbf{I}_d\bigotimes\hat{\mathbf{D}}(G)-\sum_{g\in\Gamma}\rho(g)\bigotimes\mathbf{A}_g),$$

where $n=\mid V(G)\mid$, $m=\mid E(G)\mid$ and $D(G)=\{e_1^{\pm1},\dots,e_m^{\pm1}\}$.

Proof. Let $V(G)=\{v_1,\cdots,v_n\}$ and, let $D(G)=\{e_1,\cdots,e_m,e_1^{-1},\cdots,e_m^{-1}\}$. Arrange arcs of G as follows: $e_1,e_1^{-1},\cdots,e_m,e_m^{-1}$. Then two $bd\times bd$ matrices $\mathbf{B}^\rho_w=(\mathbf{B}^{\rho,w}_{e,f})_{e,f\in D(G)}$ and $\mathbf{J}^\rho_w=(\mathbf{J}^{\rho,w}_{e,f})_{e,f\in D(G)}$ are defined as follows:

$$\mathbf{B}^{\rho,w}_{e,f}=\begin{cases}\rho(\alpha(e))\bigotimes\mathbf{w}(e) & \text{if } t(e)=o(f),\\ \mathbf{0}_{da_{t(e)},da_{o(f)}} & \text{otherwise,}\end{cases}\quad \mathbf{J}^{\rho,w}_{e,f}=\begin{cases}\rho(\alpha(e))\bigotimes\mathbf{w}(e) & \text{if } f=e^{-1},\\ \mathbf{0}_{da_{t(e)},da_{o(f)}} & \text{otherwise.}\end{cases}$$

Now, set $D(G)=\{e_1,\cdots,e_m,e_{m+1},\cdots,e_{2m}\}$ such that $e_{m+i}=e_i^{-1}$ $(1\le i\le m)$. For each arc $e_j\in D(G)$ $(1\le j\le 2m)$, let $\mathbf{X}_{e_j}$ be the $b\times b$ matrix whose the $dc_{j-1}+1,\cdots,dc_j$ rows are the $dc_{j-1}+1,\cdots,dc_j$ rows of $\mathbf{B}^\rho_w-\mathbf{J}^\rho_w$, and whose other rows are $\mathbf{0}$, where $c_j=\sum_{k=1}^j a_{o(e_k)}$ for $1\le i\le m$. Set $\mathbf{M}=\mathbf{I}-\sum_{e\in D(G)}\mathbf{X}_e$. Then, for any sequence of arcs π,

$$\det(\mathbf{I}-\mathbf{X}_\pi)=\begin{cases}\det(\mathbf{I}-\rho(\alpha(\pi))\bigotimes\mathbf{w}(\pi)) & \text{if } \pi \text{ is a prime, reduced cycle,}\\ 1 & \text{otherwise,}\end{cases}$$

where $\mathbf{X}_\pi=\mathbf{X}_{e_1}\cdots\mathbf{X}_{e_r}$ for $\pi=(e_1\cdots e_r)$. By Theorem 3, we have

$$\zeta_G(\mathbf{w},\rho,\alpha)^{-1}=\det\mathbf{M}=\det(\mathbf{I}_{bd}-(\mathbf{B}^\rho_w-\mathbf{J}^\rho_w)).$$

But, we have

$$\mathbf{U}_{\rho}\mathbf{B}_{\rho}=\mathbf{B}^{\rho}_{w} \textit{ and } \mathbf{U}_{\rho}\mathbf{J}_{\rho}=\mathbf{J}^{\rho}_{w}.$$

Thus,

$$\mathbf{B}^{\rho}_{w}-\mathbf{J}^{\rho}_{w}=\mathbf{U}_{\rho}(\mathbf{B}_{\rho}-\mathbf{J}_{\rho}).$$

Therefore, it follows that

$$\zeta_G(\mathbf{w},\rho,\alpha)^{-1}=\det(\mathbf{I}_{bd}-\mathbf{U}_{\rho}(\mathbf{B}_{\rho}-\mathbf{J}_{\rho})).$$

Now, let $\mathbf{K}_{\rho}=(\mathbf{K}^{(\rho)}_{ev})_{\ e\in D(G);v\in V(G)}$ be the $bd\times ad$ matrix defined as follows:

$$\mathbf{K}^{(\rho)}_{ev}:=\begin{cases}\mathbf{I}_d\bigotimes\mathbf{I}_{a_v} & \text{if } o(e)=v,\\ \mathbf{0}_{da_{o(e)},da_v} & \text{otherwise.}\end{cases}$$

Furthermore, we define the $bd\times ad$ matrix $\mathbf{L}=(\mathbf{L}^{(\rho)}_{ev})_{e\in D(G);v\in V(G)}$ as follows:

$$\mathbf{L}^{(\rho)}_{ev}:=\begin{cases}\mathbf{I}_d\bigotimes\mathbf{I}_{a_v} & \text{if } t(e)=v,\\ \mathbf{0}_{da_{t(e)},da_v} & \text{otherwise.}\end{cases}$$

Then we have

$${\mathbf{L}_{\rho}}^t\mathbf{K}_{\rho}=\mathbf{B}_{\rho}.$$

Thus,

$$\begin{aligned}&\det(\mathbf{I}_{bd}-\mathbf{U}_{\rho}(\mathbf{B}_{\rho}-\mathbf{J}_{\rho}))\\ =\ &\det(\mathbf{I}_{bd}-\mathbf{U}_{\rho}({\mathbf{L}_{\rho}}^t\mathbf{K}_{\rho}-\mathbf{J}_{\rho}))=\det(\mathbf{I}_{bd}-\mathbf{U}_{\rho}{\mathbf{L}_{\rho}}^t\mathbf{K}_{\rho}+\mathbf{U}_{\rho}\mathbf{J}_{\rho}).\end{aligned}$$

Now, the $bd\times ad$ matrix $\mathbf{U}_{\rho}\mathbf{L}_{\rho}=(c^{(\rho)}_{ev})_{e\in D(G);v\in V(G)}$ is given as follows:

$$c^{(\rho)}_{ev}:=\begin{cases}\rho(\alpha(e))\bigotimes\mathbf{w}(e) & \text{if } t(e)=v,\\ \mathbf{0}_{da_{t(e)},da_v} & \text{otherwise.}\end{cases}$$

Furthermore, we have

$$\begin{aligned}&\det\left(\begin{bmatrix}\mathbf{I}&\mathbf{F}\\ \mathbf{H}&\mathbf{I}\end{bmatrix}\right)=\det\left(\begin{bmatrix}\mathbf{I}&\mathbf{F}\\ \mathbf{H}&\mathbf{I}\end{bmatrix}\right)\cdot\det\left(\begin{bmatrix}\mathbf{I}&\mathbf{0}\\ -\mathbf{H}&\mathbf{I}\end{bmatrix}\right)\\ =\ &\det\left(\begin{bmatrix}\mathbf{I}-\mathbf{F}\mathbf{H}&\mathbf{F}\\ \mathbf{0}&\mathbf{I}\end{bmatrix}\right)=\det(\mathbf{I}-\mathbf{F}\mathbf{H}),\end{aligned}$$

and so,

$$\begin{aligned}\det(\mathbf{I}_{bd}+\mathbf{U}_{\rho}\mathbf{J}_{\rho})&=\prod^{m}_{i=1}\det(\mathbf{I}_{da_{o(e_i)}}-(\rho(\alpha(e_i))\bigotimes\mathbf{w}(e_i))(\rho(\alpha(e_i^{-1}))\bigotimes\mathbf{w}(e_i^{-1}))\\ &=\prod^{m}_{i=1}\det(\mathbf{I}_{da_{o(e_i)}}-\mathbf{I}_d\bigotimes\mathbf{w}(e_i)\mathbf{w}(e_i^{-1}))\\ &=\prod^{m_1}_{i=1}\det(\mathbf{I}_d)^{a_{o(e_i)}}\det(\mathbf{I}_{a_{o(e_i)}}-\mathbf{w}(e_i)\mathbf{w}(e_i^{-1}))^d\\ &=\prod^{m}_{i=1}\det(\mathbf{I}_{a_{o(e_i)}}-\mathbf{w}(e_i)\mathbf{w}(e_i^{-1}))^d.\end{aligned}$$

By the hypothesis that

$$\det(\mathbf{I}_{a_{o(e)}}-\mathbf{w}(e)\mathbf{w}(e^{-1}))\neq 0$$

for each $e\in D(G)$, we have

$$\det(\mathbf{I}_{bd}+\mathbf{U}_{\rho}\mathbf{J}_{\rho})\neq 0.$$

Thus, $\mathbf{I}_{bd} + \mathbf{U}_\rho \mathbf{J}_\rho$ is invertible. Therefore, it follows that

$$\det(\mathbf{I}_{bd} - \mathbf{U}_\rho(\mathbf{B}_\rho - \mathbf{J}_\rho)) = \det(\mathbf{I}_{bd} - \mathbf{U}_\rho \mathbf{L}_\rho {}^t\mathbf{K}_\rho(\mathbf{I}_{bd} + \mathbf{U}_\rho \mathbf{J}_\rho)^{-1}) \det(\mathbf{I}_{bd} + \mathbf{U}_\rho \mathbf{J}_\rho).$$

But, if $\mathbf{A}$ and $\mathbf{B}$ are a $p \times q$ and $q \times p$ matrices, respectively, then we have

$$\det(\mathbf{I}_p - \mathbf{AB}) = \det(\mathbf{I}_q - \mathbf{BA}).$$

Thus, we have

$$\begin{aligned} & \det(\mathbf{I}_{bd} - \mathbf{U}_\rho(\mathbf{B}_\rho - \mathbf{J}_\rho)) \\ = \ & \det(\mathbf{I}_{ad} - {}^t\mathbf{K}_\rho(\mathbf{I}_{bd} + \mathbf{U}_\rho \mathbf{J}_\rho)^{-1} \mathbf{U}_\rho \mathbf{L}_\rho) \det(\mathbf{I}_{bd} + \mathbf{U}_\rho \mathbf{J}_\rho). \end{aligned}$$

Next, we have

$$\begin{bmatrix} \mathbf{I} & \mathbf{F} \\ \mathbf{H} & \mathbf{I} \end{bmatrix}^{-1} = \begin{bmatrix} (\mathbf{I} - \mathbf{FH})^{-1} & -\mathbf{F}(\mathbf{I} - \mathbf{HF})^{-1} \\ -\mathbf{H}(\mathbf{I} - \mathbf{FH})^{-1} & (\mathbf{I} - \mathbf{HF})^{-1} \end{bmatrix}.$$

Thus, we have

$$(\mathbf{I}_{bd} + \mathbf{U}_\rho \mathbf{J}_\rho)^{-1} = \begin{bmatrix} (\mathbf{I}_d \bigotimes (\mathbf{I} - \mathbf{w}(e_1)\mathbf{w}(e_1^{-1})))^{-1} & -(\rho(\alpha(e_1)) \bigotimes \mathbf{w}(e_1))(\mathbf{I}_d \bigotimes (\mathbf{I} - \mathbf{w}(e_1^{-1})\mathbf{w}(e_1)))^{-1} & \cdots \\ -(\rho(\alpha(e_1^{-1})) \bigotimes \mathbf{w}(e_1^{-1}))(\mathbf{I}_d \bigotimes (\mathbf{I} - \mathbf{w}(e_1)\mathbf{w}(e_1^{-1})))^{-1} & (\mathbf{I}_d \bigotimes (\mathbf{I} - \mathbf{w}(e_1^{-1})\mathbf{w}(e_1)))^{-1} & \\ \vdots & & \ddots \end{bmatrix}$$

Since $(\mathbf{I} \bigotimes \mathbf{F})(\mathbf{I} \bigotimes \mathbf{H})^{-1} = \mathbf{I} \bigotimes \mathbf{FH}^{-1}$, we have

$$(\mathbf{I}_{bd} + \mathbf{U}_\rho \mathbf{J}_\rho)^{-1} = \begin{bmatrix} \mathbf{I}_d \bigotimes x_1^{-1} & -\rho(\alpha(e_1)) \bigotimes \mathbf{w}(e_1) y_1^{-1} & \\ -\rho(\alpha(e_1^{-1})) \bigotimes \mathbf{w}(e_1^{-1}) x_1^{-1} & \mathbf{I}_d \bigotimes y_1^{-1} & \\ & & \ddots \end{bmatrix},$$

where $x_i = \mathbf{I}_{a_{o(e_i)}} - \mathbf{w}(e_i)\mathbf{w}(e_i^{-1})$ and $y_i = \mathbf{I}_{a_{t(e_i)}} - \mathbf{w}(e_i^{-1})\mathbf{w}(e_i)$ for $1 \leq i \leq m$.

But, for an arc $(x, y) \in D(G)$,

$$({}^t\mathbf{K}_\rho(\mathbf{I}_{bd} + \mathbf{U}_\rho \mathbf{J}_\rho)^{-1} \mathbf{U}_\rho \mathbf{L}_\rho)_{xy} = \rho(\alpha(x, y)) \bigotimes (\mathbf{I}_{a_x} - \mathbf{w}(x, y)\mathbf{w}(y, x))^{-1} \mathbf{w}(x, y).$$

Furthermore, if $x = y$, then

$$({}^t\mathbf{K}_\rho(\mathbf{I}_{bd} + \mathbf{U}_\rho \mathbf{J}_\rho)^{-1} \mathbf{U}_\rho \mathbf{L}_\rho)_{xx} = -\mathbf{I}_d \bigotimes \sum_{o(e)=x} \mathbf{w}(e)(\mathbf{I}_{a_{t(e)}} - \mathbf{w}(e^{-1})\mathbf{w}(e))^{-1} \mathbf{w}(e^{-1}).$$

Thus,

$$\det(\mathbf{I}_{ad} - {}^t\mathbf{K}_\rho(\mathbf{I}_{bd} + \mathbf{U}_\rho \mathbf{J}_\rho)^{-1} \mathbf{U}_\rho \mathbf{L}_\rho) = \det(\mathbf{I}_{ad} + \mathbf{I}_d \bigotimes \hat{\mathbf{D}}(G) - \sum_{g \in \Gamma} \rho(g) \bigotimes \mathbf{A}_g),$$

Therefore, it follows that

$$\zeta_G(\mathbf{w}, \rho, \alpha)^{-1} = \prod_{i=1}^{m} \det(\mathbf{I}_{a_{o(e_i)}} - \mathbf{w}(e_i)\mathbf{w}(e_i^{-1}))^d \det(\mathbf{I}_{ad} + \mathbf{I}_d \bigotimes \widehat{\mathbf{D}}(G) - \sum_{g \in \Gamma} \rho(g) \bigotimes \mathbf{A}_g).$$

Q.E.D.

By Theorems 4,5, the following result holds.

Corollary 1 *Let G be a connected graph, Γ a finite group, $\alpha : D(G) \longrightarrow \Gamma$ an ordinary voltage assignment and $\{\mathbf{w}(e) \mid e \in D(G)\}$ a matrix-weight of G. Then we have*

$$\zeta_{G^\alpha}(\tilde{\mathbf{w}}) = \prod_\rho \zeta_G(\mathbf{w}, \rho, \alpha)^{\deg \rho},$$

where ρ runs over all inequivalent irreducible representations of Γ.

4 Special case

Let G be a connected graph, $a_v \in \mathbf{N}$ for each $v \in V(G)$, and $\{\mathbf{w}(e) \mid e \in D(G)\}$ a matrix-weight of G, where $\mathbf{w}(e)$ is an $a_{o(e)} \times a_{t(e)}$ matrix.

Now, suppose that $a_v = a$ (constant) $\in \mathbf{N}$ for any $v \in V(G)$ and $\mathbf{w}(e^{-1}) = \mathbf{w}(e)^{-1}$ for each $e \in D(G)$. Then we define a special matrix-weighted zeta function of G as follows:

$$\zeta_G(\mathbf{w}, t) = \prod_{[C]} (1 - \mathbf{w}(C) t^{|C|})^{-1},$$

where $[C]$ runs over all equivalence classes of prime, reduced cycles of G.

Considering the matrix-weighted zeta function of a graph G for the matrix-weight $\{\mathbf{w}(e)t \mid e \in D(G)\}$, we obtain a determinant expression for $\zeta_G(\mathbf{w}, t)$.

Corollary 2 *Let G be a connected graph with n vertices and m edges, $a \in \mathbf{N}$, and $\{\mathbf{w}(e)t \mid e \in D(G)\}$ a matrix-weight of G, where $\mathbf{w}(e)$ is an $a \times a$ matrix.*

Then the reciprocal of $\zeta_G(\mathbf{w}, t)$ for G is given by

$$\zeta_G(\mathbf{w}, t)^{-1} = (1 - t^2)^{(m-n)a} \det(\mathbf{I}_{an} - t\mathbf{W}(G) + t^2(\mathbf{D} - \mathbf{I}_n) \bigotimes \mathbf{I}_a),$$

where the $an \times an$ matrix $\mathbf{W}(G) = (w_{xy})_{x,y \in V(G)}$ is given as follows:

$$w_{xy} = \begin{cases} \mathbf{w}(x, y) & \text{if } (x, y) \in D(G), \\ \mathbf{0}_a & \text{otherwise.} \end{cases}$$

Proof. Since $\mathbf{w}(e^{-1}) = \mathbf{w}(e)^{-1}$ for each $e \in D(G)$, we have $\mathbf{w}(e)\mathbf{w}(e^{-1}) = \mathbf{w}(e^{-1})\mathbf{w}(e) = \mathbf{I}_a$. By Theorem 2, we have

$$\zeta_G(\mathbf{w}, t)^{-1} = \prod_{i=1}^m \det(\mathbf{I}_a - t^2 \mathbf{I}_a) \det(\mathbf{I}_{an} + \hat{\mathbf{D}}(G) - \hat{\mathbf{A}}).$$

But, we have

$$\hat{\mathbf{D}} = \frac{t^2}{1 - t^2} \mathbf{D} \bigotimes \mathbf{I}_a, \ \hat{\mathbf{A}} = \frac{t}{1 - t^2} \mathbf{W}(G).$$

Thus,

$$\zeta_G(\mathbf{w}, t)^{-1} = (1 - t^2)^{ma} \det(\mathbf{I}_{an} - t/(1 - t^2)\mathbf{W}(G) + t^2/(1 - t^2)\mathbf{D} \bigotimes \mathbf{I}_a)$$

$$= \ (1 - t^2)^{(m-n)a} \det(\mathbf{I}_{an} - t\mathbf{W}(G) + t^2(\mathbf{D} - \mathbf{I}_n) \bigotimes \mathbf{I}_a).$$

Q.E.D.

In the case of $a = 1$, the zeta function $\zeta_G(w, t)$ of G is the weighted zeta function $\mathbf{Z}(G, w, t)$ of G introduced by Mizuno and Sato [8]:

$$\mathbf{Z}(G, w, t) = \prod_{[C]} (1 - w(C) t^{|C|})^{-1},$$

where $[C]$ runs over all equivalence classes of prime, reduced cycles of G.

By Corollary 2, we obtain a determinant expression for the weighted zeta function of a graph.

Corollary 3 (Mizuno and Sato) *Let G be a connected graph with n vertices and m edges, $\{w(e)t \mid e \in D(G)\}$ a scalar-weight of G. Suppose that $w(e^{-1}) = w(e)^{-1}$ for each $e \in D(G)$. Then the reciprocal of $\mathbf{Z}(G, w, t)$ is given by*

$$\mathbf{Z}(G, w, t)^{-1} = (1 - t^2)^{m-n} \det(\mathbf{I}_n - t\mathbf{W}(G) + t^2(\mathbf{D} - \mathbf{I}_n)),$$

where $\mathbf{W}(G)$ is an $n \times n$ matrix.

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