Completely Regular Clique Graphs*

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1 Introduction

In this article, I introduce completely regular clique graphs. After reviewing basic properties of distance-regular completely regular clique graphs in [22, 23], I will present a recent result in [25] showing that majority of known distance-regular graphs of large diameter are completely regular clique graphs. In addition, I will explain why I am interested in distance-regular completely regular clique graphs. I also add open problems on completely regular clique graphs.

Throughout, we consider only finite graphs. Let $\Gamma = (X, R)$ be a connected graph with vertex set X and edge set R consisting of 2-element subsets of X. When $\{x, y\} \in R$, i.e., x and y are adjacent, we write $x \sim y$. For $x, y \in X$, $\partial_\Gamma(x, y) = \partial(x, y)$ denotes the distance between x and y, i.e., the length of a shortest path between x and y in Γ. The diameter $d(\Gamma)$ is the maximal distance between two vertices. For $C \subset X$, C is also regarded as an induced subgraph of Γ. A nonempty subset C of X is said to be a clique if every distinct vertices in C are adjacent.

A nonempty subset C of X is often called a *code* in $\Gamma = (X, R)$. For each integer i, the subset $\Gamma_i(C) = \{x \in X \mid \partial(x, C) = i\}$ is called the *ith subconstituent* with respect to C, where $\partial(x, C) = \min\{\partial(x, y) \mid y \in C\}$. We write $\Gamma(C)$ for $\Gamma_1(C)$. The number $\delta = \delta(C) = \max\{i \mid \Gamma_i(C) \neq \emptyset\}$ is called the *covering radius* of C. When $C = \{x\}$, we write $\Gamma_i(x)$ for $\Gamma_i(\{x\})$, and set $\Gamma(x)$ for $\Gamma_1(x)$. The number $k(x) = |\Gamma(x)|$ is called the *valency* of x. For $x, y \in X$ with $\partial(x, y) = i$ with $i \in \{0, 1, \ldots, d(\Gamma)\}$, let

$$B_i(x, y) = \Gamma_{i+1}(x) \cap \Gamma(y),\ A_i(x, y) = \Gamma_i(x) \cap \Gamma(y),\ C_i(x, y) = \Gamma_{i-1}(x) \cap \Gamma(y),$$

and $b_i(x, y) = |B_i(x, y)|$, $a_i(x, y) = |A_i(x, y)|$ and $c_i(x, y) = |C_i(x, y)|$.

A connected graph $\Gamma = (X, R)$ of diameter $d = d(\Gamma)$ is said to be *distance-regular* if, for each $i \in \{0, 1, 2, \ldots, d\}$, the numbers $c_i = c_i(x, y)$, $a_i = a_i(x, y)$ and $b_i = b_i(x, y)$

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depend only on $i = \partial(x, y)$. For distance-regular graphs we refer the reader to [6]. We mainly follow the notation and the terminologies in the monograph.

Let $\Gamma = (X \cup Y, R)$ be a connected bipartite graph with the bipartition $X \cup Y$, i.e., there is no edge within X, and Y. Let $d^X = d^X(\Gamma) = \max\{\partial(x, y) \mid x \in X, y \in X \cup Y\}$. Then Γ is said to be *distance-semiregular* on X, if for each $i \in \{0, 1, 2, \ldots, d^X\}$, the numbers $c_i^X = c_i(x, y)$, and $b_i^X = b_i(x, y)$ depend only on $i = \partial(x, y)$ whenever $x \in X$ and $y \in X \cup Y$. Note that each vertex $y \in Y$ is of valency $b_0^Y = b_1^X + c_1^X$ and distance-semiregular graphs are biregular, i.e., the valency of a vertex depends only on the part the vertex belongs to. If $\Gamma = (X \cup Y, R)$ is distance-semiregular on both X and Y, Γ is called *distance-biregular*. For more information on distance-biregular graphs and distance-semiregular graphs, see [21].

For a bipartite graph $\tilde{\Gamma} = (X \cup Y, \tilde{R})$ with the bipartition $X \cup Y$, the *bipartite half of* $\tilde{\Gamma}$ *on* X is a graph with vertex set X such that two vertices are adjacent whenever they are at distance 2 in $\tilde{\Gamma}$. It is easy to see that if $\tilde{\Gamma} = (X \cup Y, \tilde{R})$ is a distance-semiregular graph on X, then its bipartite half on X is distance-regular.

Let $\Gamma = (X, R)$ be a connected graph, and C a nonempty subset of X with covering radius $\delta = \delta(C)$. Then C is said to be a *completely regular code* if $\gamma_i = \gamma_i(x) = |\Gamma_{i-1}(C) \cap \Gamma(x)|$, $\alpha_i = \alpha_i(x) = |\Gamma_i(C) \cap \Gamma(x)|$, $\beta_i = \beta_i(x) = |\Gamma_{i+1}(C) \cap \Gamma(x)|$ do not depend on $x \in \Gamma_i(C)$ for $i \in \{0, 1, \ldots, \delta\}$. For completely regular codes of distance-regular graphs, see [6, Section 11.1] and [20].

Definition 1.1 Let $\Gamma = (X, R)$ be a connected graph, and let $\mathcal{C}$ be a collection of cliques of Γ. Then Γ is said to be a *completely regular clique graph with parameters* (s, c) *with respect to* $\mathcal{C}$, if the following are satisfied.

(i) Each member $C \in \mathcal{C}$ is a completely regular code of size $s + 1 \geq 2$.

(ii) Each edge is contained in exactly c members of $\mathcal{C}$ and $c \geq 1$.

When $\mathcal{C}$ consists of Delsarte cliques, it is called a Delsarte clique graph with parameters (s, c) in [1, 2], and Delsarte clique graphs with parameters $(s, 1)$ are called geometric in [3]. For examples of Delsarte clique graphs, see [1].

Before we state basic properties of completely regular clique graphs, we define regular incidence structures as follows.

Definition 1.2 Let $\mathcal{I} = (X, Y, I)$ be an incidence structure, where X and Y are nonempty finite sets, and I a subset of $X \times Y$, i.e., a relation between X and Y. When $(x, y) \in I$, we write xIy, and we say that x is incident with y, y is incident with x, or x and y are incident. Then $\mathcal{I}$ is said to be a *regular incident structure with parameters* (s, t, c) with $s, t, c \geq 1$, if the following are satisfied.

(i) Each element $x \in X$ is incident with exactly $t + 1$ elements of Y.

(ii) Each element $y \in Y$ is incident with exactly $s + 1$ elements of X.

(iii) For every pair of distinct elements $y, y' \in Y$, there is $x \in X$ such that $(x, y) \in I$ and $(x, y') \notin I$.

(iv) For every pair of distinct elements $x, x' \in X$,

$$|\{y \in Y \mid (x, y) \in I \text{ and } (x', y) \in I\}| \in \{0, c\}.$$

Let $\mathcal{I} = (X, Y, I)$ be an incidence structure.

The *collinearity graph* $\Gamma = (X, R_I)$ of $\mathcal{I}$ is a graph with vertex set X and edge set $R_I = \{\{x, x'\} \subset X \mid x \neq x' \text{ and there exists } y \in Y \text{ such that } (x, y) \in I \text{ and } (x', y) \in I\}$.

The *incidence graph* $\tilde{\Gamma} = (X \cup Y, \tilde{R}_I)$ is a bipartite graph with vertex set $X \cup Y$ and edge set $\tilde{R}_I = \{\{x, y\} \mid x \in X, y \in Y, (x, y) \in I\}$.

An incidence structure $\mathcal{I} = (X, Y, I)$ is said to be *connected* if its incidence graph is connected.

Let $\mathcal{I} = (X, Y, I)$ be a regular incidence structure. Then it is clear from our definition that $\mathcal{I}$ is connected if and only if its collinearity graph is connected. For each $y \in Y$, let $C_y = \{x \in X \mid (x, y) \in I\}$. Then by the condition (iii), $C_y \neq C_{y'}$ if $y, y' \in Y$ are distinct.

Let $\tilde{\Gamma} = (X \cup Y, \tilde{R})$ be a distance-semiregular graph on X with the following property.

$$\text{For } y, y' \in Y \text{ with } y \neq y',\ \tilde{\Gamma}(y) \neq \tilde{\Gamma}(y'). \tag{1}$$

Then $\tilde{\Gamma}$ is the incidence graph of a connected regular incidence structure.

We now list known basic properties of completely regular clique graphs shown in [22, 23].

1. Let Γ be a completely regular clique graph of parameters (s, c) with respect to a collection $\mathcal{C}$ of cliques.

   (a) Each member $C \in \mathcal{C}$ is a maximal clique if $d(\Gamma) > 1$. (See a remark preceding Lemma 8 in [23].)

   (b) The parameters of completely regular codes $C \in \mathcal{C}$ do not depend on C. (Theorem 1 in [22])

   (c) Γ is regular, and by setting $Y = \mathcal{C}$ and for $x \in X$ and $C \in Y$, xIC if and only if $x \in C$, $\mathcal{I} = (X, Y, I)$ defines a connected regular incidence structure with parameters $(s, ck/s - 1, c)$, where k is the valency of Γ. The collinearity graph of $\mathcal{I}$ is Γ. (See Lemma 9 in [22], and Section 2.1 in [23].)

2. Let $\mathcal{I} = (X, Y, I)$ be a connected regular incidence structure with parameters (s, t, c). Set $\mathcal{C} = \{C_y \mid y \in Y\}$, where $C_y = \{x \in X \mid (x, y) \in I\}$. Then the following are equivalent.

   (i) The collinearity graph $\Gamma = (X, R_I)$ of $\mathcal{I}$ is a distance-regular completely regular clique graph of parameters (s, c) with respect to $\mathcal{C}$.

(ii) The incidence graph $\tilde{\Gamma} = (X \cup Y, \tilde{R}_I)$ of $\mathcal{I}$ is distance-semiregular on X.

(See Theorem 3, Propositions 8, 10 in [22] and Theorem 3 in [23].)

3. Every bipartite distance-regular graph of diameter at least three satisfies the condition (1). In particular, the bipartite half of a bipartite distance-regular graph of diameter at least three is a completely regular clique graph.

2 My Personal Historical Remarks

Let me explain how I became interested in completely regular clique graphs.

Recall that a connected regular and edge regular graph is said to be of order (s,t) if it does not contain $K_{2,1,1}$. It is easy to see that for such a graph there are nonnegative integers s and t such that $\Gamma(x)$ is a $t+1$ disjoint union of cliques of size s. Hence the valency is $s(t+1)$. Moreover, every maximal clique is of size $s+1$ and each edge is contained in exactly one such maximal clique. Therefore, every distance-regular graph of order (s,t) affords a regular incidence structure of parameters $(s,t,1)$.

In [18], with A. Hiraki and K. Nomura, we determined all distance-regular graphs of valency 6 and $a_1 = 1$. It is easy to see that if a distance-regular graph satisfies $c_2 = 1$ or $a_1 \leq 1$, then it is of order (s,t) for some integers s and t. Hence the graphs in question are of order $(2,2)$ and afford regular incidence structures of parameters $(2,2,1)$. By our investigation we could show that the incidence graph of distance-regular graph of order $(2,2)$ enjoy high regularity. The bipartite graphs with this high regularity were later defined as distance-semiregular graphs in [21]. We knew that the incidence graphs of many distance-regular graphs of order (s,t) are distance-semiregular, but we were interested in when the incidence graph of a distance-regular graph of order (s,t) is distance-semiregular as most of our works then were focused on it.

In [21], besides defining the concept of distance-semiregular graphs, I showed that distance-regular graphs of order (s,t) with $s > t$ have Delsarte cliques and the incidence graphs are distance-semiregular. For a distance-regular graph which is obtained as the bipartite half of a distance-semiregular graph, I showed that many parametrical restrictions which do not hold in general.

In [24], I studied the Terwilliger algebra $\mathcal{T}(C)$ of a distance-regular graph with respect to a subset C, and if a vector satisfies a special condition, called *tight*, it generates a thin irreducible module, i.e., a module on which the adjacency matrix acts as a tridiagonal linear transformation. In addition, if the base subset C is completely regular, the primary module is thin, and if C is a clique, every non-primary irreducible $\mathcal{T}(C)$-module of endpoint zero, i.e., the module containing a vector whose support intersects with C, is thin. Since every Delsarte clique is completely regular, I thought that most of the results for Delsarte cliques may well be generalized to completely regular cliques.

Thus, when J. Koolen and others started to study Delsarte clique graphs in [1, 2, 3], I made a plan to investigate Delsarte clique graphs replacing Delsarte cliques by completely regular cliques. I knew that there are distance-regular graphs which do not have Delsarte clique graphs, but have good collections of completely regular cliques.

In 2012, I received a question from M. Fiol by email, and started a collaboration with his group. We showed that for a connected graph, every edge is completely regular with same parameters if and only if it is an almost bipartite or bipartite distance-regular graph in [7].

Soon after this, I started to study completely regular clique graphs, and showed that the parameters are uniquely determined if the size of completely regular cliques is fixed. See [22]. Thus the result in [7] holds even if we do not assume a condition on parameters. This was a breakthrough because the condition on parameters is easily obtained algebraically for Delsarte cliques, but we did not know a corresponding result for completely regular cliques.

There was another motivation. Brouwer-Wilbrink, Brouwer-Cohen, and de Bruyn [5, 4, 11] classified regular near polygons of diameter at least three with $c_2 > 2$. For the classification of large class of distance-regular graphs. I am convinced that the techniques of incidence geometries are very powerful, and the study of universal covers is the key when we want to recognize the structure locally. The vertex-clique incidence graph of a distance-regular graph, when we consider it as a point-line geometry, is a starting point. Every distance-regular completely regular clique graph affords a certain incidence structure with high regularity, and I believe this is a very good class of distance-regular graphs to study using techniques of incidence geometry. The collaboration with B. de Bruyn in [12] helped my understanding.

Let Γ be a distance-regular completely regular clique graph with parameters (s, c) of valency $k > 2$ and diameter at least two. If $s = 1$, then it is an edge distance-regular graph, and Γ is either bipartite or almost bipartite and $c = 1$. Hence the distance-2-graph of Γ is a distance-regular completely regular clique graph with parameters (s', c') with $s = k - 1 > 1$ and $c' = c_2$ of Γ. Therefore, if we focus on distance-regular completely regular clique graphs of parameters (s, c), we may assume that $s > 1$, i.e., so-called thick line case.

3 Completely Regular Clique Graphs of Known-Type

We state our main theorem. The definitions of the graphs in the theorem are found in [6].

Theorem 3.1 *Let Γ be one of the distance-regular graphs listed below:*

> *The polygons, the Johnson graphs, the folded Johnson graphs, the Odd graphs, the doubled Odd graphs, the Grassmann graphs, the doubled Grassmann graphs, the Hamming graphs, the halved hypercubes, the folded hypercubes, the halved folded hypercubes, the dual polar graphs, the half dual polar graphs of type D, the Ustimenko graphs, the Hemmeter graphs, the bilinear forms graphs, the alternating forms graphs, and the quadratic forms graphs.*

Then Γ *is a completely regular clique graph.*

The Doob graphs, the twisted Grassmann graphs, and the Hermitean forms graphs of diameter at least three are not completely regular clique graphs with respect to any collection of cliques.

We also investigated the collections of cliques $\mathcal{C}$, the graph becomes a completely regular clique graph with respect to $\mathcal{C}$.

Since the bipartite half of a bipartite distance-regular graph is a completely regular clique graph, Theorem 3.1 can be considered as a generalization of the works of J. Hemmeter and others who determined bipartite distance-regular graphs whose bipartite half are known distance-regular graphs of large diameter. See [13, 14, 15, 16, 17].

Corollary 3.2 *Let* Γ *be one of the distance-regular graphs listed in Theorem 3.1. If* Γ *is a bipartite half of a distance-regular graph* $\tilde{\Gamma}$*, then* $(\Gamma, \tilde{\Gamma})$ *is one of the following:*
$(J(2N+1, N), \textit{Doubled Odd Graph})$, $(J_q(2N+1, N), \textit{Doubled Grassmann Graph})$, $(\frac{1}{2}H(N,2), H(N,2))$, $(\frac{1}{2}\overline{H(2N,2)}, \overline{H(2N,2)})$, $(\frac{1}{2}(\textit{DPG of type D}), \textit{DPG of type D})$, *or* $(\textit{Ustimenko Graph}, \textit{Hemmeter Graph})$.

4 Two Technical Results

To prove Theorem 3.1, we use the information on the complete list of maximal cliques obtained by J. Hemmeter and others.

When the graph is known to be a completely regular clique graph with respect to a collection of cliques, the following result is useful to search another collection of cliques that also defines a completely regular clique graph on the same graph.

Theorem 4.1 *Let* Γ *be a distance-regular graph of valency* k *and diameter* $d \geq 2$*, and let* $\mathcal{C}$ *and* $\mathcal{C}'$ *be collections of cliques. Suppose* Γ *is a completely regular clique graph with respect to both* $\mathcal{C}$ *and* $\mathcal{C}'$ *with parameters* (s, c) *and* (s', c') *respectively. Assume that there are* $C \in \mathcal{C}$ *and* $C' \in \mathcal{C}'$ *such that* $|C \cap C'| = e > 1$*. If* $s \neq s'$*, then*

$$ss' = k(e-1).$$

Our proof of Theorem 4.1 uses the primary modules of Terwilliger algebras with respect to $C \in \mathcal{C}$ and $C' \in \mathcal{C}'$. I do not know if the result can be proved by combinatorial argument.

If Γ is a distance-regular graph of order (s, t), the result follows easily. Note that the Hermitean forms graphs are distance-regular graphs of order (s, t) but each maximal clique is not completely regular if the diameter is at least three.

Theorem 4.2 *Let* Γ *be a distance-regular graph of order* (s, t)*. Then the following are equivalent.*

(i) Γ *is a completely regular clique graph.*

(ii) Γ *is a completely regular clique graph of parameters* $(s, 1)$ *with respect to the collection of all maximal cliques.*

(iii) *For every maximal clique* C, C *is a completely regular code.*

Proof. By the the definition of completely regular clique graphs, the equivalence of (i), (ii) and (iii) is obvious. ■

Corollary 4.3 *The following graphs are completely regular clique graphs with respect to the collection of all maximal cliques, and there is no other choice of a collection of maximal cliques these graphs afford a structure of a completely regular clique graph.*

(i) *The Hamming graphs* $H(d, q)$.

(ii) *The dual polar graphs:*

$$C_d(q),\ B_d(q),\ D_d(q),\ {}^2D_{d+1}(q),\ {}^2A_{2d}(r),\ {}^2A_{2d-1}(r) \text{ with } q = r^2.$$

(iii) *Bipartite distance-regular graphs and almost bipartite distance-regular graphs: The polygons, the Odd graphs, the doubled Odd graphs, the doubled Grassmann graphs, the folded hypercubes, the Hemmeter graphs.*

Proof. These graphs are known to be of order (s, t). See [6]. ■

5 Concluding Remarks

In this section, we collect problems and remarks.

Problem 1 Are completely regular clique graphs distance-regular?

Problem 2 Is there a distance-regular graph of diameter $d \geq 2$ which is of completely regular clique graph with respect to three collections of cliques of different size?

By Theorem 4.1, if Γ is a completely regular clique graph with respect to two collections of cliques of different size s and s', then $ss' = k(e-1)$. Hence, if s is fixed, e is determined by s'.

When Γ is the half dual polar graph of type D, the Ustimenko graph, the alternating forms graph, or the quadratic forms graph, we failed to determine the parameter c, and hence $\mathcal{C}$ completely.

Problem 3 Determine possible values of c, i.e., the collection $\mathcal{C}$ of cliques in the remaining cases.

The incidence graph $\tilde{\Gamma}$ of a completely regular clique graph Γ with respect to a collection of cliques $\mathcal{C}$ is often realized as an induced subgraph on $\Delta_{m-1}(x) \cup \Delta_m(x)$ of a bipartite distance-regular graph Δ, and in several cases $m = d(\Delta)$.

In [8], Caughman proved that if Δ is a Q-polynomial bipartite distance-regular graph of diameter $m = d(\Delta)$, then the distance-2-graph Γ on $\Delta_m(x)$ is a Q-polynomial distance-regular graph for each vertex x. For example, if Δ is the dual polar graph $D_d(q)$, then Γ is isomorphic to $\mathrm{Alt}_q(n)$. See [6, Proposition 9.5.11]. If Δ is the Hemmeter graph $\mathrm{Hem}_d(q)$ with q odd, then Γ is isomorphic to $\mathrm{Quad}_q(d)$. What happens if Γ is isomorphic to $\mathrm{Quad}_q(d)$ with q even? See [9, 19].

Problem 4 Which completely regular clique graph Γ with respect to a collection of cliques $\mathcal{C}$ can be realized as an induced subgraph of the distance-2-graph on the last subconstituent of a bipartite distance-regular graph Δ?

Problem 5 Classify distance-regular graphs cospectral to the dual polar graphs of type D. Is there a distance-regular graph whose last subconstituent is isomorphic to $\mathrm{Quad}_q(d)$ with q even?

In [21], we showed several nice parametrical conditions of distance-semiregular graphs. Hence distance-regular completely regular clique graphs have higher regularity that general distance-regular graphs do not enjoy.

Problem 6 Is there an absolute bound on the girth of the incidence graph of a completely regular clique graph of parameters (s, c) with $s > 1$?

In Table 1, we summarize Theorem 3.1 for the case $s > 1$. Note that

$$\mu_i = |C \cap \Gamma_i(x)|, \text{ where } x \in \Gamma_i(C),$$

for $i \in \{0, 1, \ldots, \delta(C)\}$ and $C \in \mathcal{C}$. By a result in [20], if C is a clique of a distance-regular graph Γ, C is completely regular if and only if μ_i depends only on i, and the parameters are determined by μ_i's and the parameters of Γ.

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Table 1: Known DR CRCGs of Unbounded Diam. ($s > 1$, $d \geq 3$)

Γ	**Graph**	d	$s+1$	c	μ_i
$J(n,m)$	Johnson graph, $n \geq 2m$	m	$n-m+1$	1	$i+1$
$J(n,m)$	Johnson graph, $n \geq 2m$	m	$m+1$	1	$i+1$
$J(2m,m)$	Johnson graph,	m	$m+1$	2	$i+1$[a]
$\overline{J(2m,m)}$	Folded Johnson	$[m/2]$	$m+1$	2	$i+1$
$J_q(n,m)$	Grassmann graph, $n \geq 2m$	m	$[n-m+1]_q$	1	$[i+1]_q$
$J_q(n,m)$	Grassmann graph, $n \geq 2m$	m	$[m+1]_q$	1	$[i+1]_q$
$J_q(2m,m)$	Grassmann graph	m	$[m+1]_q$	2	$[i+1]_q$
$H(n,m)$	Hamming graph, $m \geq 3$	n	m	1	1
$\frac{1}{2}H(n,2)$	Halved hypercube	$[n/2]$	n	1	$2i+1$
$\frac{1}{2}\overline{H(2n,2)}$	Halved folded hypercube	$[n/2]$	$2n$	2	$2i+1$[b]
$C_n(q)$	Dual polar graph of type C	n	$q+1$	1	1
$B_n(q)$	Dual polar graph of type B	n	$q+1$	1	1
${}^2D_{n+1}(q)$	Dual polar graph of type ${}^2D$	n	q^2+1	1	1
${}^2A_{2n}(r)$	Dual polar graph of type ${}^2A$	n	r^3+1	1	1
${}^2A_{2n-1}(r)$	Dual polar graph of type ${}^2A$	n	$r+1$	1	1
$\frac{1}{2}D_n(q)$	Half dual polar graph of type D	$[n/2]$	$[n]_q$	$\leq [2]_q$	$[2i+1]_q$
$\frac{1}{2}\mathrm{Hem}_n(q)$	Ustimenko graph	$[n/2]$	$[n]_q$	$\leq [2]_q$	$[2i+1]_q$
$\mathrm{Bilin}_q(n,m)$	Bilinear forms graph, $n \geq m$	m	q^n	1	1
$\mathrm{Bilin}_q(n,m)$	Bilinear forms graph, $n \geq m$	m	q^m	1	1
$\mathrm{Bilin}_q(m,m)$	Bilinear forms graph	m	q^m	2	1
$\mathrm{Alt}_q(n)$	Alternating forms graph	$[n/2]$	q^n	$\leq [2]_q$	q^{2i}
$\mathrm{Quad}_q(n)$	Quadratic forms graph	$[(n+1)/2]$	q^n	$\leq [2]_q$	q^{2i}

a: If $m = 2d+1$ is odd, $\mu_d = (2d+1)^2 = m^2$.
b: If $n = 2d+1$ is odd, $\mu_d = 4d+2 = 2n$.

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